

# $\mathbb{P}^1$ -BUNDLES OVER PROJECTIVE MANIFOLDS OF PICARD NUMBER ONE WHICH ADMIT ANOTHER SMOOTH MORPHISM OF RELATIVE DIMENSION ONE

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ABSTRACT. We give a complete classification of  $\mathbb{P}^1$ -bundles over a projective manifold of Picard number one which admit another smooth morphism of relative dimension one.

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## 1. INTRODUCTION

R. Muñoz, G. Occhetta and L. Solá Conde studied rank 2 vector bundles on Fano manifolds in [11]. In their paper, they obtained a complete list of  $\mathbb{P}^1$ -bundles over a Fano manifold with  $b_2 = b_4 = 1$  that have a second  $\mathbb{P}^1$ -bundle structure. The purpose of this paper is to generalize the result. Actually, we give a complete classification of  $\mathbb{P}^1$ -bundles over a projective manifold of Picard number 1 which admit another smooth morphism of relative dimension 1. Our main result is the following:

**Theorem 1.1.** *Let  $X$  be a complex projective manifold of Picard number  $\rho = 1$  and  $\mathcal{E}$  a rank 2 vector bundle on  $X$ . Assume that  $Z := \mathbb{P}(\mathcal{E}) \rightarrow X$  admits another smooth morphism  $Z \rightarrow Y$  of relative dimension 1 and  $n := \dim X \geq 2$ . Then,*

- (I)  *$X$  and  $Y$  are Fano manifolds of  $\rho = 1$  and there exists a rank 2 vector bundle  $\mathcal{E}'$  on  $Y$  such that  $Z \rightarrow Y$  is given by  $\mathbb{P}_Y(\mathcal{E}')$ .*

Furthermore,

- (II) *if  $\mathcal{E}$  and  $\mathcal{E}'$  are normalized by twisting with line bundles (i.e.,  $c_1 = 0$  or  $-1$ ), then, up to changing the pairs  $(X, \mathcal{E})$  and  $(Y, \mathcal{E}')$ ,  $((X, \mathcal{E}), (Y, \mathcal{E}'))$  is one of the following:*
- (a)  *$((\mathbb{P}^2, T_{\mathbb{P}^2}), (\mathbb{P}^2, T_{\mathbb{P}^2}))$ , where  $T_{\mathbb{P}^2}$  is the tangent bundle of the projective plane  $\mathbb{P}^2$ ,*

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- (b)  $((\mathbb{P}^3, \mathcal{N}), (Q^3, \mathcal{S}))$ , where  $\mathcal{N}$  is the null-correlation bundle on  $\mathbb{P}^3$  (see [13]) and  $\mathcal{S}$  is the restriction to the 3-dimensional quadric  $Q^3$  of the universal quotient bundle from the Grassmannian  $G(1, \mathbb{P}^3)$ ,
- (c)  $((Q^5, \mathcal{C}), (K(G_2), \mathcal{Q}))$ , where  $\mathcal{C}$  is a Cayley bundle on  $Q^5$  (see [14]),  $K(G_2)$  is a 5-dimensional Fano homogeneous contact manifold of type  $G_2$  which is a linear section of the Grassmannian  $G(1, \mathbb{P}^6)$  and  $\mathcal{Q}$  the restriction of the universal quotient bundle on  $G(1, \mathbb{P}^6)$ .

Consequently,  $Z$  is the full-flag manifold of type  $A_2$ ,  $B_2$  or  $G_2$ . In particular,  $X$ ,  $Y$  and  $Z$  are rational homogeneous manifolds.

On the other hand, Campana and Peternell proposed the following conjecture:

**Conjecture 1.2** ([3]). *A Fano manifold  $M$  with nef tangent bundle is homogeneous.*

This conjecture is true in dimension  $\leq 4$ . The most difficulty lies in the case where  $M$  is a Fano 4-fold of  $\rho = 1$  which carries a rational curve  $C$  with  $-K_M.C = 3$ . In this case, N. Mok [10] proved the conjecture under the additional assumption that  $b_4(M) = 1$ . After that, J. M. Hwang pointed out that the assumption  $b_4(M) = 1$  can be removed in [7]. More generally, they obtained

**Theorem 1.3** ([10, Main Theorem], [7]). *Let  $M$  be a Fano manifold of  $\rho = 1$  with nef tangent bundle. Assume that  $M$  carries a rational curve  $C$  such that  $-K_M.C = 3$ . Then  $M$  is isomorphic to  $\mathbb{P}^2$ ,  $Q^3$  or  $K(G_2)$ .*

In their proof, it is crucial to investigate a family  $X$  of minimal rational curves on  $M$  and its universal family  $Z$  which satisfies the assumption as in Theorem 1.1 (when we regard  $M$  as  $Y$  in 1.1). From this viewpoint, we can consider Theorem 1.1 as a generalization of Theorem 1.3. Remark that we cannot prove Theorem 1.1 by the same method as in [10], since it is not necessary that  $X$  can be regarded as a minimal rational component of  $Y$ . For instance, in the case where  $(X, Y) \cong (Q^3, \mathbb{P}^3)$ ,  $Q^3$  is not a minimal rational component of  $\mathbb{P}^3$ .

The contents of this paper are organized as follows. Section 2 is devoted to study the structures of the Chow group  $A_2(X)_{\mathbb{Q}}$  of 2-dimensional cycles with  $\mathbb{Q}$ -coefficients and its quotient  $N^2(X)_{\mathbb{Q}}$  by numerical equivalence, according to a similar argument as in [7]. In Section 3, we give two computations of the discriminant  $\Delta(\mathcal{E}) := c_1^2(\mathcal{E}) - 4c_2(\mathcal{E})$  which are based on ideas in [10] and [11] (see Proposition 3.6 and 3.12). In Section 4, by comparing the computational results, we narrow down the possible values of some invariants of  $X$  and  $Y$ . Then we can show the existence of a rank 2 vector bundle  $\mathcal{E}'$  on  $Y$  such that  $Z \rightarrow Y$  is given by  $\mathbb{P}_Y(\mathcal{E}')$  (Proposition 4.5). It turns out that  $Z$  admits double  $\mathbb{P}^1$ -bundle structures  $\pi : Z \rightarrow X$  and  $\phi : Z \rightarrow Y$ . Then the same argument as in [11] implies Theorem 1.1 (Theorem 4.6).

In this paper, we use notation as in [5] and every point on a variety we deal with is a closed point. We work over the field of complex numbers.

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## 2. STRUCTURES OF $A_2(X)_{\mathbb{Q}}$ AND $N^2(X)_{\mathbb{Q}}$

First, we prove a part of Theorem 1.1 (I). That is

**Proposition 2.1.** *Let  $X$  be a projective manifold of  $\rho_X = 1$  and  $\mathcal{E}$  a rank 2 vector bundle on  $X$ . Assume that  $\pi : Z = \mathbb{P}(\mathcal{E}) \rightarrow X$  admits another smooth morphism  $\phi : Z \rightarrow Y$  of relative dimension 1 and  $n := \dim X \geq 2$ . Then*

- (i)  $Y$  is a Fano manifold of  $\rho_Y = 1$ ,
- (ii)  $\phi^{-1}(y)$  is isomorphic to  $\mathbb{P}^1$  for every  $y \in Y$ , and
- (iii)  $X$  is also a Fano manifold of  $\rho_X = 1$ .

(i) is easy to see. In fact, since  $\rho_Z = 2$ , it turns out that  $\rho_Y = 1$ . Additionally,  $Y$  is covered by rational curves. Thus  $Y$  is a Fano manifold of  $\rho_Y = 1$ .

We use the following lemma to prove the second part of Proposition 2.1.

**Lemma 2.2.** *Let  $X$  be a projective manifold and  $Y$  a Fano manifold of  $\rho = 1$ . Assume that a projective manifold  $Z$  admits two different smooth morphisms  $\pi : Z \rightarrow X$  and  $\phi : Z \rightarrow Y$  of relative dimension 1 and  $\pi^{-1}(x)$  is isomorphic to  $\mathbb{P}^1$  for every  $x \in X$ . Given  $y \in Y$ , define inductively*

- (i)  $V_y^0 := \{y\}$ , and
- (ii)  $V_y^{m+1} := \phi(\pi^{-1}(\pi(\phi^{-1}(V_y^m))))$ .

*Then there exists a natural number  $l$  such that  $V_y^l = Y$ .*

*Proof.* The idea of this proof is in [9]. Since  $V_y^k$  is an irreducible closed subset of  $Y$ , it is sufficient to show that  $V_y^k = Y$  provided  $\dim V_y^k = \dim V_y^{k+1}$ . Remark that  $\dim V_y^k$  is independent of the choice of  $y \in Y$ . It follows from flatness of  $\pi$  and  $\phi$ . Assume that  $\dim V_y^k = \dim V_y^{k+1}$  for any  $y \in Y$ . Then  $V_y^k = V_y^{k+1}$ . Supposing that  $V_y^k$  does not coincide with  $Y$ , we shall derive a contradiction. Let  $q$  be the codimension of  $V_y^k$  in  $Y$  and  $T \subset Y$  a  $(q-1)$ -dimensional projective subvariety. From our assumption, we have  $q \geq 1$ . Denote  $\bigcup_{y \in T} V_y^k$  by  $A$ . Since  $\rho_Y = 1$ ,  $A$  is an ample divisor on  $Y$ . Hence, for any point  $x \in X$ ,  $\phi(\pi^{-1}(x)) \cap A \neq \emptyset$ , then there exists a point  $y_x \in T$  such that  $\phi(\pi^{-1}(x)) \cap V_{y_x}^k \neq \emptyset$ . This implies that  $\phi(\pi^{-1}(x))$  is contained in  $V_{y_x}^{k+1} = V_{y_x}^k \subset A$ . However this contradicts the surjectivity of  $\phi$ .  $\square$

*Proof of Proposition 2.1(ii) and (iii).* Assume that there exists a fiber of  $\phi$  which is not rational. Then every fiber of  $\phi$  is not rational. Let  $f$  be a fiber of  $\pi$  and  $\nu$  the restriction of  $\phi$  to  $f \cong \mathbb{P}^1$ . Consider a smooth family of curves  $\nu^*Z \rightarrow f \cong \mathbb{P}^1$ . Since a fiber of  $\nu^*Z \rightarrow f \cong \mathbb{P}^1$  is not rational, the family is isotrivial. Furthermore, the family is trivial by virtue of the simply-connectedness of  $\mathbb{P}^1$ . It turns out that  $\pi(\phi^{-1}(y_1)) = \pi(\phi^{-1}(y_2))$  for any  $y_1, y_2 \in \phi(\pi^{-1}(x))$  provided we fix a point  $x \in X$ . From Lemma 2.2, it follows that any two point can be connected by a chain of rational curves  $\phi(\pi^{-1}(x))$  of finite length. Hence we see that  $\pi(\phi^{-1}(y_1)) = \pi(\phi^{-1}(y_2))$  for any  $y_1, y_2 \in Y$ . However, this is a contradiction to the surjectivity of  $\pi$ . As a consequence, every fiber of  $\phi$  is rational. Furthermore,  $X$  is covered by rational curves and  $\rho_X = 1$ . It turns out that  $X$  is a Fano manifold of  $\rho_X = 1$ .  $\square$

Next, we prove the following:

**Proposition 2.3.** *Let  $X$  be an  $n$ -dimensional Fano manifold of  $\rho = 1$  and  $\mathcal{E}$  a rank 2 vector bundle on  $X$ . Assume that  $Z := \mathbb{P}(\mathcal{E}) \rightarrow X$  admits another smooth*

morphism  $\phi : Z \rightarrow Y$  whose fiber is isomorphic to  $\mathbb{P}^1$ . If  $n \geq 2$ , then  $A_2(X)_{\mathbb{Q}}$  and  $N^2(X)_{\mathbb{Q}}$  are isomorphic to the field of rational numbers  $\mathbb{Q}$ .

**Definition 2.4.** For a point  $y \in Y$ , define inductively

- (i)  $W_y^0 := \phi^{-1}(y)$ ,  $\widetilde{W}_y^0 := W_y^0 \times_X Z$  and
- (ii)  $W_y^k := \widetilde{W}_y^{k-1} \times_Y Z$ ,  $\widetilde{W}_y^k := W_y^k \times_X Z$ .

Remark that  $\widetilde{W}_y^{k-1}$  has a natural morphism to  $Y$  defined by the composition of a projection  $\widetilde{W}_y^{k-1} \rightarrow Z$  and  $\phi : Z \rightarrow Y$ . On the other hand,  $W_y^k$  admits a natural morphism to  $X$  by the composition of a projection  $W_y^k \rightarrow Z$  and  $\pi : Z \rightarrow X$ . Hence we can define  $W_y^k$  and  $\widetilde{W}_y^k$  as above.

**Remark 2.5.** For a point  $y \in Y$ , the image of the composition of a projection  $\widetilde{W}_y^k \rightarrow Z$  and  $\phi : Z \rightarrow Y$  coincides with  $V_y^{k+1}$  as in Lemma 2.2.

**Lemma 2.6.** Let  $p : W' \rightarrow W$  be a  $\mathbb{P}^1$ -bundle with a section  $\sigma : W \rightarrow W'$ . Then any  $\gamma \in A_k(W')$  is of the form  $\gamma = \sigma_*\alpha + p^*\beta$  for some  $\alpha \in A_k(W)$  and  $\beta \in A_{k-1}(W)$ .

*Proof.* See [4, Theorem 3.3]. □

*Proof of Proposition 2.3.* By Lemma 2.2, there exists  $l \in \mathbb{N}$  such that  $\widetilde{W}_y^{l-1} \rightarrow Y$  is surjective. Hence  $W_y^l \rightarrow X$  is also surjective. Then, so is  $A_2(W_y^l)_{\mathbb{Q}} \rightarrow A_2(X)_{\mathbb{Q}}$ . Thus, to prove  $A_2(X)_{\mathbb{Q}} \cong \mathbb{Q}$ , we show that the rank of  $A_2(W_y^l)_{\mathbb{Q}} \rightarrow A_2(X)_{\mathbb{Q}}$  is at most 1. Since  $W_y^k$  and  $\widetilde{W}_y^k$  are rationally connected,  $A_0(W_y^k)$  and  $A_0(\widetilde{W}_y^k)$  are isomorphic to the ring of integers  $\mathbb{Z}$ . Then it follows from Lemma 2.6 that  $A_1(W_y^k)$  and  $A_1(\widetilde{W}_y^k)$  are generated by curves whose images in  $Z$  are either a fiber of  $\pi$  or a fiber of  $\phi$ . Furthermore,  $A_2(W_y^k)$  and  $A_2(\widetilde{W}_y^k)$  are generated by surfaces whose images in  $Z$  are either a curve or surfaces of the form  $\phi^{-1}(\phi(C))$  for some fiber  $C$  of  $\pi$  or  $\pi^{-1}(\pi(C'))$  for some fiber  $C'$  of  $\phi$ . This concludes that  $A_2(X)_{\mathbb{Q}}$  is isomorphic to  $\mathbb{Q}$ . Since  $A_2(X)_{\mathbb{Q}} \rightarrow N_2(X)_{\mathbb{Q}}$  is surjective,  $N_2(X)_{\mathbb{Q}}$  is also isomorphic to  $\mathbb{Q}$ . Thus we obtain  $N^2(X)_{\mathbb{Q}} \cong \mathbb{Q}$ . □

### 3. COMPUTATION OF THE DISCRIMINANT $\Delta(\mathcal{E})$

From now on, we work under the following notation and assumption:

**Notation-Assumptions 3.1.** Let  $X$  be an  $n$ -dimensional Fano manifold of  $\rho = 1$  and  $\mathcal{E}$  a rank 2 vector bundle on  $X$ . Denote the natural projection by  $\pi : Z := \mathbb{P}(\mathcal{E}) \rightarrow X$ . Assume that  $n \geq 2$  and  $Z$  admits another smooth morphism  $\phi : Z \rightarrow Y$  whose fiber is isomorphic to  $\mathbb{P}^1$ . Since  $X$  and  $Y$  are Fano manifolds of  $\rho = 1$ ,  $\text{Pic}(X)$  and  $\text{Pic}(Y)$  are isomorphic to  $\mathbb{Z}$ . These ample generators denote  $H_X$  and  $H_Y$ , respectively. We denote the Fano indices of  $X$  and  $Y$  by  $i_X$  and  $i_Y$ , respectively. We may assume that  $\mathcal{E}$  is normalized, i.e.,  $c_1 := c_1(\mathcal{E})$  is equal to 0 or  $-1$  (when the Picard group of  $X$  is identified with  $\mathbb{Z}$ ). From the assumption that  $n \geq 2$ , it follows that  $\mathcal{E}$  is not trivial. Furthermore, we fix notation as follows:  $H := \pi^*H_X$ ,  $H' := \phi^*H_Y$ ,  $d_X := H_X^n$ ,  $d_Y := H_Y^n$ ,  $\mu := H \cdot f'$ ,  $\mu' := H' \cdot f$ ,  $\Delta(\mathcal{E}) := c_1^2(\mathcal{E}) - 4c_2(\mathcal{E})$ , where  $f$  and  $f'$  are fibers of  $\pi$  and  $\phi$ , respectively. From Proposition 2.3,  $N^2(X)_{\mathbb{Q}} = \mathbb{Q}\Sigma$

for a positive cycle  $\Sigma$  on  $X$  of codimension 2. Furthermore rational numbers  $c_2, d$  and  $\Delta$  are defined by equations  $c_2(\mathcal{E}) =: c_2\Sigma, H_X^2 =: d\Sigma, \Delta(\mathcal{E}) =: (d\Delta)\Sigma$ .  $K_\pi$  and  $L$  stand for the relative canonical divisor and a divisor associated with the tautological line bundle of  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ , respectively. Then we define  $\tau := \tau(\mathcal{E})$  as the only real number such that  $-K_\pi + \tau H$  is nef but not ample and  $v := v(\mathcal{E})$  as the only real number such that  $-K_\pi + vH$  is pseudoeffective but not big.

First, we review the definition of (semi)stability for vector bundles and some results in [11].

**Definition 3.2.** Under the same setting as in Notation-Assumptions 3.1, let  $A$  be an ample divisor on  $X$ . Then  $\mathcal{E}$  is said to be *stable* (resp. *semistable*) if, for any line bundle  $L \subset \mathcal{E}$ ,

$$c_1(L).A^{n-1} < \frac{1}{2}c_1(\mathcal{E}).A^{n-1} \text{ (resp. } c_1(L).A^{n-1} \leq \frac{1}{2}c_1(\mathcal{E}).A^{n-1}).$$

**Theorem 3.3.** *Let  $(X, \mathcal{E})$  be as in Notation-Assumptions 3.1. If  $\mathcal{E}$  is semistable, then, for an ample divisor  $A \in \text{Pic}(X)$ , we have*

$$\Delta(\mathcal{E}).A^{n-2} \leq 0.$$

*Proof.* This follows from Bogomolov inequality and Mehta-Ramanathan theorem.  $\square$

**Theorem 3.4** ([11, Theorem 2.3, Proposition 3.5, Remark 3.6]). *Let  $(X, \mathcal{E})$  be as in Notation-Assumptions 3.1. Then the following holds.*

- (i)  $\tau \geq 0$ , and equality holds if and only if  $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$ .
- (ii) If  $\mathcal{E}$  is not semistable, then  $v < 0$ .
- (iii) If  $\mathcal{E}$  is semistable but not stable, then  $v = 0$ .

Thanks to Proposition 2.3, the same argument as in [11] can be applied to our case. In particular, we obtain the following Proposition 3.5 and 3.6. For the readers convenience, we recall their argument.

**Proposition 3.5** (cf. [11, Proposition 4.12]). *Under the setting as in Notation-Assumptions 3.1,*

- (i)  $\tau = v = i_X - \frac{2}{\mu} \in \mathbb{Q}_{>0}$ , and
- (ii)  $\mathcal{E}$  is stable.

*Proof.* (i) Since  $\rho_Z = 2$ , the Kleiman-Mori cone of  $Z$  is spanned by  $f$  and  $f'$ . Furthermore, we have  $-K_Z.f = -K_Z.f' = 2$ . By Kleiman's criterion for ampleness, this implies that  $-K_Z$  is ample, that is,  $Z$  is a Fano manifold. So the nef cone of  $Z$  is rational polyhedral. This implies that  $\tau$  is a rational number. It follows from Base-point-free theorem that  $-K_\pi + \tau H$  is semiample. Then it turns out that  $\phi$  is defined by the linear system  $|m(-K_\pi + \tau H)|$  if  $m$  is sufficiently large and divisible. This implies that  $(-K_\pi + \tau H).f' = 0$ . Thus we see that  $\tau = i_X - \frac{2}{\mu}$ . Furthermore, since  $\phi$  is a morphism of relative dimension 1,  $-K_\pi + \tau H$  is nef but not big. Hence we get  $\tau = v$ . If  $\tau = 0$ , then  $\mathcal{E}$  is trivial by Theorem 3.4 (i). Hence  $\tau > 0$ .

(ii) By Theorem 3.4 (ii), we see that  $\mathcal{E}$  is semistable. If  $\mathcal{E}$  is semistable but not stable, then it follows from Theorem 3.4 (iii) that  $\tau = v = 0$ . This is a contradiction.  $\square$

**Proposition 3.6** (cf. [11, Proposition 4.4]). *Under the setting as in Notation-Assumptions 3.1,*

- (i)  $\Delta < 0$ ,
- (ii)  $\sqrt{-\Delta} = \tau \tan(\frac{\pi}{n+1})$ , and
- (iii)  $n = 2, 3$  or  $5$ .

*Proof.* (i) By Proposition 3.5,  $\mathcal{E}$  is stable and  $-K_\pi + \tau H$  is nef but not big. So we have  $\Delta \leq 0$  and  $(-K_\pi + \tau H)^{n+1} = 0$ . Since  $K_\pi^2 = \Delta H^2$ ,  $H^{n+1} = 0$  and  $-K_\pi \cdot H^n > 0$ , we get

$$(1) \quad \sum_{\substack{i=0 \\ i \equiv 1(2)}}^{n+1} \binom{n+1}{i} \tau^{n+1-i} \Delta^{\frac{i-1}{2}} = 0.$$

If  $\Delta = 0$ , then  $\tau^n = 0$ . It means that  $\tau = 0$ . However, again by Theorem 3.4, it implies that  $\mathcal{E}$  is trivial. This is a contradiction. As a consequence, we have  $\Delta < 0$ .

(ii) From the above equation (1), we obtain

$$(2) \quad (\tau + \sqrt{\Delta})^{n+1} - (\tau - \sqrt{\Delta})^{n+1} = 0.$$

We denote the argument of the complex number  $\tau + \sqrt{\Delta}$  by  $\arg(\tau + \sqrt{\Delta}) \in [0, 2\pi)$ . Then (2) is equivalent to

$$(3) \quad \arg(\tau + \sqrt{\Delta}) = 0 \text{ or } \frac{\pi}{n+1}.$$

Since we have  $\Delta < 0$  by (i), (3) implies

$$\sqrt{-\Delta} = \tau \tan\left(\frac{\pi}{n+1}\right).$$

(iii) From (ii), we obtain

$$\tan^2\left(\frac{\pi}{n+1}\right) = \frac{-\Delta}{\tau^2} \in \mathbb{Q}.$$

The algebraic degree of  $\tan\left(\frac{\pi}{n+1}\right)$  over  $\mathbb{Q}$  is known (see [12, pp. 33-41] and [2, Proposition 2]). Then we see that  $n = 2, 3$  or  $5$ . □

On the other hand, we give another description of  $\Delta(\mathcal{E})$  via a computation of the total Chern class  $c(\pi^*\mathcal{E})$ . First, we prepare the following lemma.

**Lemma 3.7.** *Under the setting as in Notation-Assumptions 3.1, let  $\sigma$  denote the restriction of  $\pi$  to  $f' \cong \mathbb{P}^1$ . If  $\sigma^*\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$  ( $a \geq b$ ), then we have*

$$(a, b) = \left(-1 + \frac{(c_1 + i_X)\mu}{2}, 1 + \frac{(c_1 - i_X)\mu}{2}\right).$$

*Proof.* Let consider a  $\mathbb{P}^1$ -bundle  $\sigma^*Z \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$  over  $f' \cong \mathbb{P}^1$ . Then  $\sigma^*f'$  is an exceptional curve on  $\sigma^*Z$ . It implies that  $\sigma^*f' \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(b))$ . Hence  $b = L \cdot f' = 1 + \frac{(c_1 - i_X)\mu}{2}$ . On the other hand, we have  $a + b = c_1\mu$ . Thus  $a = -1 + \frac{(c_1 + i_X)\mu}{2}$ . □

**Lemma 3.8.** *Under the setting as in Notation-Assumptions 3.1, the total Chern class  $c(\pi^*\mathcal{E})$  is given by*

$$c(\pi^*\mathcal{E}) = 1 + \frac{1}{\mu}(a+b)H + \left( \frac{ab}{\mu^2}H^2 + \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2 \right).$$

*Proof.* Let  $P$  be the kernel of  $\pi^*\mathcal{E} \rightarrow L$ . Then we have an exact sequence

$$0 \rightarrow P \rightarrow \pi^*\mathcal{E} \rightarrow L \rightarrow 0.$$

In general, any saturated subsheaf of a locally-free sheaf is again locally-free. So  $P$  is a line bundle. Since  $L|_f \cong \mathcal{O}_{\mathbb{P}^1}(1)$  and  $L|_{f'} \cong \mathcal{O}_{\mathbb{P}^1}(b)$ , we see that  $P|_f = \ker(\pi^*\mathcal{E}|_f \rightarrow \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$  and  $P|_{f'} = \ker(\pi^*\mathcal{E}|_{f'} \rightarrow \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathcal{O}_{\mathbb{P}^1}(a)$ . Remark that  $A^1(Z)_{\mathbb{Q}} = \langle H, H' \rangle_{\mathbb{Q}}$ . Hence we obtain

$$\begin{aligned} c(L) &= 1 + \frac{b}{\mu}H + \frac{1}{\mu'}H', \\ c(P) &= 1 + \frac{a}{\mu}H - \frac{1}{\mu'}H', \end{aligned}$$

and

$$\begin{aligned} c(\pi^*\mathcal{E}) &= c(L) \cdot c(P) = \left( 1 + \frac{b}{\mu}H + \frac{1}{\mu'}H' \right) \cdot \left( 1 + \frac{a}{\mu}H - \frac{1}{\mu'}H' \right) \\ &= 1 + \frac{1}{\mu}(a+b)H + \left( \frac{ab}{\mu^2}H^2 + \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2 \right). \end{aligned}$$

□

**Lemma 3.9.** *Under the setting as in Notation-Assumptions 3.1, let  $\nu$  be the restriction of  $\phi$  to  $f \cong \mathbb{P}^1$  and  $\zeta$  a projection  $\nu^*Z \rightarrow Z$ . If  $N_{\nu^*f/\nu^*Z} \cong \mathcal{O}_{\mathbb{P}^1}(-e)$ , then  $e > 0$  and we have*

$$(\zeta^*H)^2 = \frac{\mu e}{\mu'}(\zeta^*H\zeta^*H').$$

*Proof.* We consider a  $\mathbb{P}^1$ -bundle  $\psi : \nu^*Z \rightarrow f \cong \mathbb{P}^1$ . Then  $\nu^*f$  is an exceptional curve on  $\nu^*Z$ . From  $N_{\nu^*f/\nu^*Z} \cong \mathcal{O}_{\mathbb{P}^1}(-e)$ , we obtain  $e > 0$ . Furthermore, we see that  $\nu^*Z \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$  and  $\nu^*f \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-e))$ . Let  $M$  be the tautological line bundle of  $\nu^*Z \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$  and  $Q$  the kernel of  $\psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \rightarrow M$ . Then we have an exact sequence

$$0 \rightarrow Q \rightarrow \psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \rightarrow M \rightarrow 0.$$

Then we see that  $M|_{\nu^*f} \cong \mathcal{O}_{\mathbb{P}^1}(-e)$  and  $M|_{\nu^*f'} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ . These imply that

$$\begin{aligned} Q|_{\nu^*f} &= \ker(\psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))|_{\nu^*f} \rightarrow \mathcal{O}_{\mathbb{P}^1}(-e)) \cong \mathcal{O}_{\mathbb{P}^1}, \text{ and} \\ Q|_{\nu^*f'} &= \ker(\psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))|_{\nu^*f'} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

Remark that  $A^1(\nu^*Z)_{\mathbb{Q}} = \langle \nu^*H, \nu^*H' \rangle_{\mathbb{Q}}$ . Hence we obtain

$$\begin{aligned} c(M) &= 1 + \frac{1}{\mu}\zeta^*H - \frac{e}{\mu'}\zeta^*H', \\ c(Q) &= 1 - \frac{1}{\mu}\zeta^*H. \end{aligned}$$

and

$$\begin{aligned} c(\psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))) &= c(M) \cdot c(Q) = \left(1 + \frac{1}{\mu}\zeta^*H - \frac{e}{\mu'}\zeta^*H'\right) \cdot \left(1 - \frac{1}{\mu}\zeta^*H\right) \\ &= 1 - \frac{e}{\mu'}\zeta^*H' + \left(-\frac{1}{\mu^2}(\zeta^*H)^2 + \frac{e}{\mu\mu'}\zeta^*H\zeta^*H'\right). \end{aligned}$$

Furthermore, we obtain

$$-\frac{1}{\mu^2}(\zeta^*H)^2 + \frac{e}{\mu\mu'}\zeta^*H\zeta^*H' = c_2(\psi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))) = \psi^*(c_2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))) = 0.$$

As a consequence, we get

$$(\zeta^*H)^2 = \frac{\mu e}{\mu'}(\zeta^*H\zeta^*H')$$

as desired.  $\square$

**Lemma 3.10.** *Under the setting as in Notation-Assumptions 3.1, we have*

$$\frac{a-b}{e\mu^2}H^2 = \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2 \in N^2(Z)_{\mathbb{Q}}.$$

*Proof.* In Lemma 3.8, we have seen that  $\pi^*(c_2(\mathcal{E})) = c_2(\pi^*\mathcal{E}) = \frac{ab}{\mu^2}H^2 + \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2$ . Since we have  $N^2(X)_{\mathbb{Q}} \cong \mathbb{Q}$ , there exists  $g \in \mathbb{Q}$  such that

$$(4) \quad gH^2 + \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2 = 0 \in N^2(Z)_{\mathbb{Q}}.$$

Pulling back to  $\nu^*Z$  by  $\zeta$ , we obtain

$$\zeta^*(gH^2 + \frac{a-b}{\mu\mu'}HH') = \zeta^*\left(gH^2 + \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2\right) = 0 \in N^2(\nu^*Z)_{\mathbb{Q}}.$$

By Lemma 3.9,  $(\zeta^*H)^2 = \frac{\mu e}{\mu'}(\zeta^*H\zeta^*H')$ . It turns out that

$$\left(\frac{g\mu e}{\mu'} + \frac{a-b}{\mu\mu'}\right)\zeta^*H\zeta^*H' = 0 \in N^2(\nu^*Z)_{\mathbb{Q}}.$$

Hence we have  $g = \frac{b-a}{\mu^2 e}$ . Substituting this in the equation (4), we obtain

$$\frac{a-b}{e\mu^2}H^2 = \frac{a-b}{\mu\mu'}HH' - \frac{1}{\mu'^2}H'^2 \in N^2(Z)_{\mathbb{Q}}.$$

$\square$

By combining Lemma 3.8 and Lemma 3.10, we get the following:

**Proposition 3.11.** *Under the setting as in Notation-Assumptions 3.1, the total Chern class  $c(\pi^*\mathcal{E})$  is given by*

$$c(\pi^*\mathcal{E}) = 1 + \frac{1}{\mu}(a+b)H + \left(\frac{ab}{\mu^2} + \frac{a-b}{e\mu^2}\right)H^2 \in 1 \oplus N^1(Z) \oplus N^2(Z)_{\mathbb{Q}}.$$

**Proposition 3.12.** *Under the setting as in Notation-Assumptions 3.1,*

$$\Delta = \tau^2 - \frac{4\tau}{e\mu}.$$



*Proof.* From the definition of  $\Delta$  and Proposition 3.11,

$$\begin{aligned}\Delta H^2 &= c_1(\pi^*\mathcal{E})^2 - 4c_2(\pi^*\mathcal{E}) \\ &= \left(\frac{1}{\mu}(a+b)H\right)^2 - 4\left(\frac{ab}{\mu^2} + \frac{a-b}{e\mu^2}\right)H^2 = \frac{a-b}{\mu^2 e}(e(a-b) - 4)H^2.\end{aligned}$$

By Lemma 3.7,  $a-b = i_X\mu - 2 = \tau\mu$ . Thus, we obtain  $\Delta = \tau^2 - \frac{4\tau}{e\mu}$  as desired.  $\square$

#### 4. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. It is sufficient to work under the same setting as in Notation-Assumptions 3.1. Then it follows from Proposition 3.6 that  $n = 2, 3$  or  $5$ .

**Theorem 4.1.** *With the same setting as in Notation-Assumptions 3.1, if  $n = 2$ , then  $(X, \mathcal{E})$  is isomorphic to  $(\mathbb{P}^2, T_{\mathbb{P}^2})$ .*

*Proof.* By virtue of Proposition 3.6 (ii) and Proposition 3.12, we have  $\Delta = -3\tau^2$  and  $\Delta = \tau^2 - \frac{4\tau}{e\mu}$ . Recall that  $\tau > 0$  by Proposition 3.5 (i). Thus  $(i_X\mu - 2)e = \tau\mu e = 1$ . On the other hand, since a Fano surface of  $\rho = 1$  is isomorphic to  $\mathbb{P}^2$ , we have  $i_X = 3$ . Hence  $(i_X, \mu, e) = (3, 1, 1)$ . Furthermore, we obtain  $\tau = 1$  and  $\Delta = -3$ . By Lemma 3.7,  $(c_1 + i_X)\mu$  is divisible by 2. This implies that  $c_1 = -1$ . Here we take a point on  $X \cong \mathbb{P}^2$  as a base  $\Sigma$  of  $N^2(X)_{\mathbb{Q}}$ . Then  $d = 1$ . Moreover, we see that  $c_2 = 1$ . Hence  $\mathcal{E}$  is a rank 2 stable vector bundle over  $\mathbb{P}^2$  with  $(c_1, c_2) = (-1, 1)$ . Then  $\mathcal{E}$  is isomorphic to  $T_{\mathbb{P}^2}$  by [6].  $\square$

**Lemma 4.2.** *With the same setting as in Notation-Assumptions 3.1, if  $n = 3$ , then  $(i_X, \mu, e) = (4, 1, 1), (3, 1, 2), (2, 2, 1), (1, 3, 2)$  or  $(1, 4, 1)$ .*

*Proof.* By virtue of Proposition 3.6 (ii) and Proposition 3.12, we have  $\Delta = -\tau^2$  and  $\Delta = \tau^2 - \frac{4\tau}{e\mu}$ . Recall that  $\tau > 0$  by Proposition 3.5 (i). Thus  $(i_X\mu - 2)e = \tau\mu e = 2$ . This implies that  $(i_X, \mu, e) = (4, 1, 1), (3, 1, 2), (2, 2, 1), (1, 3, 2)$  or  $(1, 4, 1)$ .  $\square$

**Lemma 4.3.** *Under the same setting as in Lemma 4.2, if  $(i_X, \mu, e) = (2, 2, 1)$ , then  $c_1 = 0$ .*

*Proof.* In this case,  $X$  is a del Pezzo 3-fold. So  $H^4(X, \mathbb{Z})$  is generated by a line  $l$  on  $X$ . Here a line means a rational curve with  $H_X.l = 1$ . We take  $l$  as a base  $\Sigma$  of  $N^2(X)_{\mathbb{Q}}$ . Then  $c_2$  is an integer.

Now assume the contrary of our claim, that is,  $c_1 = -1$ . Then Riemann-Roch theorem tells us that  $c_2$  is even. On the other hand, we see that  $\Delta = -1$ . Thus, we obtain  $d_X = d = 2c_2$ . From a classification of del Pezzo 3-fold of  $\rho = 1$  [8], it follows that  $d_X \leq 5$ . It turns out that  $(d_X, c_2) = (4, 2)$ . Again, according to [8],  $X$  is a complete intersection of two quadric 4-folds in  $\mathbb{P}^5$ .

Consider a morphism  $\pi \circ \zeta : \nu^*Z \rightarrow X$ . Since  $\nu^*Z \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ ,  $\pi \circ \zeta$  factors through  $g : \mathbb{P}^2 \rightarrow X$ . This can be obtained by taking the Stein factorization of  $\pi \circ \zeta = g \circ h$ , where  $h : \nu^*Z \rightarrow \mathbb{P}^2$  is a blow-up at a point  $o \in \mathbb{P}^2$ . Remark that  $h$  sends every fiber of  $\nu^*Z \rightarrow f \cong \mathbb{P}^1$  to a line through  $o \in \mathbb{P}^2$ . Hence, for a line  $l_o$  through  $o \in \mathbb{P}^2$ , we have  $g^*H_X.l_o = \mu = 2$ . This implies that  $g^*\mathcal{O}_X(H_X) \cong \mathcal{O}_{\mathbb{P}^2}(2)$ . Let  $S$  denote the image of  $g$ . Then the degree of  $S \subset \mathbb{P}^5$  satisfies that  $\deg(g)\deg(S) = 4$ . On the other hand,  $S$  is a member of the linear system  $|\mathcal{O}_X(sH_X)|$  for some  $s > 0$ . So we have  $\deg(S) = 4s$ . Hence we see that  $(s, \deg(g)) = (1, 1)$ . Then, it is easy

to see that  $S$  is smooth, that is,  $S \cong \mathbb{P}^2$ . However this contradicts the adjunction formula.  $\square$

By the same way as in Lemma 4.2, we can prove the following:

**Lemma 4.4.** *With the same setting as in Notation-Assumptions 3.1, if  $n = 5$ , then  $(i_X, \mu, e) = (5, 1, 1), (3, 1, 3), (1, 5, 1)$ , or  $(1, 3, 3)$ .*

We prove the remaining part of Theorem 1.1 (I).

**Proposition 4.5.** *Under the same setting as in Notation-Assumptions 3.1, there exists a rank 2 vector bundle  $\mathcal{E}'$  on  $Y$  such that  $\phi : Z \rightarrow Y$  is given by  $\mathbb{P}_Y(\mathcal{E}')$ .*

*Proof.* If  $n = 2$ , this follows from Theorem 4.1. Thus, we deal with the cases where  $n = 3$  and 5. From the above lemmas, it follows that

$$(n, i_X, \mu, e) = (3, 4, 1, 1), (3, 3, 1, 2), (3, 2, 2, 1), (3, 1, 3, 2), (3, 1, 4, 1), (5, 5, 1, 1), \\ (5, 3, 1, 3), (5, 1, 5, 1) \text{ or } (5, 1, 3, 3).$$

It is enough to find a line bundle  $V$  on  $Z$  which satisfies  $V.f' = 1$ . Actually,  $\mathcal{E}' := \phi_* \mathcal{O}_Z(V)$  satisfies the property desired. Hence it is sufficient to deal with the case where  $\mu := H.f' \neq 1$ , that is,  $(n, i_X, \mu, e) = (3, 2, 2, 1), (3, 1, 3, 2), (3, 1, 4, 1), (5, 1, 5, 1)$ , and  $(5, 1, 3, 3)$ .

Recall that  $L.f' = 1 + \frac{(c_1 - i_X)\mu}{2}$  due to Lemma 3.7. If  $(n, i_X, \mu, e) = (3, 2, 2, 1)$ , then, by Lemma 4.3, we have  $c_1 = 0$ . This means  $L.f' = -1$ . Hence  $V := H \otimes L$  satisfies  $V.f' = 1$ . If  $(n, i_X, \mu, e) = (3, 1, 3, 2), (5, 1, 5, 1)$  or  $(5, 1, 3, 3)$ , then we see that  $L.f' = -2, -4, -2$ , respectively. It turns out that  $V := H \otimes L$  satisfies  $V.f' = 1$ . If  $(n, i_X, \mu, e) = (3, 1, 4, 1)$ , then  $L.f' = -1$  or  $-3$ . Hence we can take  $H \otimes L^{\otimes 3}$  or  $H \otimes L$  as  $V$ .  $\square$

According to this proposition, we see that  $Z$  admits double  $\mathbb{P}^1$ -bundle structures  $\pi : Z \rightarrow X$  and  $\phi : Z \rightarrow Y$ . By symmetry of  $X$  and  $Y$ , all results on  $X$  as above also hold for  $Y$ . By the same way as in Notation-Assumptions 3.1, we define rational numbers  $c'_1, c'_2, d'$  and  $\Delta'$ . Moreover,  $K_\phi$  and  $L'$  stand for the relative canonical divisor and a divisor associated with the tautological line bundle of  $\phi : \mathbb{P}(\mathcal{E}') \rightarrow Y$ , respectively. Then we define  $\tau' := \tau(\mathcal{E}')$  and  $v' := v(\mathcal{E}')$  as in Notation-Assumptions 3.1. Applying the argument as in [11], we obtain the following:

**Theorem 4.6.** *With the same setting as in Proposition 4.5, if  $\mathcal{E}$  and  $\mathcal{E}'$  are normalized, then, up to changing the pairs  $(X, \mathcal{E})$  and  $(Y, \mathcal{E}')$ ,  $((X, \mathcal{E}), (Y, \mathcal{E}'))$  is isomorphic to  $((\mathbb{P}^3, \mathcal{N}), (Q^3, \mathcal{S}))$  or  $((Q^5, \mathcal{C}), (K(G_2), \mathcal{Q}))$ .*

*Proof.* As we have seen in Proposition 3.5,  $\tau = i_X - \frac{2}{\mu}$  and  $\tau' = i_Y - \frac{2}{\mu'}$ . Then we obtain the following table:

	$H$	$H'$	$L$	$L'$
$f$	0	$\mu'$	1	$1 + \frac{(c'_1 - i_Y)\mu'}{2}$
$f'$	$\mu$	0	$1 + \frac{(c_1 - i_X)\mu}{2}$	1

This table represents intersection numbers of a divisor and  $f$  or  $f'$ . For example,  $H.f = 0$  and  $H.f' = \mu$  etc. Applying this table, we obtain

$$(5) \quad \begin{cases} H' = -\frac{\mu'}{2}(c_1 - \tau)H + \mu' L \\ L' = \{-\frac{\mu'}{4}(c_1 - \tau)(c'_1 - \tau') + \frac{1}{\mu}\}H + \frac{\mu'}{2}(c'_1 - \tau')L. \end{cases}$$

Since  $\{H, L\}$  and  $\{H', L'\}$  are  $\mathbb{Z}$ -bases of  $\text{Pic}(\mathbb{P}(\mathcal{E}))$ , the determinant of the matrix of base change is equal to 1 or  $-1$ . This implies that  $\mu = \mu'$ . Hence we can write  $H' = \frac{\mu}{2}(-K_\pi + \tau H)$ . Furthermore, we get

$$(6) \quad \frac{d_Y}{d_X} = \left(\frac{\mu}{2}\right)^n \frac{(-K_\pi + \tau H)^n H / \mu}{-K_\pi H^n / 2} = \left(\frac{\mu}{2}\right)^{n-1} \frac{\text{im}((\tau + \sqrt{-\Delta})^n)}{\sqrt{-\Delta}}.$$

Since we have  $\sqrt{-\Delta} = \tau \tan(\frac{\pi}{n+1})$ , (6) is equivalent to

$$\frac{d_Y}{d_X} = \left(\frac{\tau \mu}{2 \cos(\pi/(n+1))}\right)^{n-1}.$$

By symmetry of  $X$  and  $Y$ , we get a similar equation

$$\frac{d_X}{d_Y} = \left(\frac{\tau' \mu'}{2 \cos(\pi/(n+1))}\right)^{n-1}.$$

These equations imply

$$(7) \quad \left(\frac{\tau \mu \tau' \mu'}{4 \cos^2(\pi/(n+1))}\right)^{n-1} = 1.$$

Since  $\tau, \tau' > 0$  and  $\mu = \mu' > 0$ , (7) provides

$$(8) \quad (i_X \mu - 2)(i_Y \mu - 2) = \tau \tau' \mu^2 = \begin{cases} 2 & (n = 3) \\ 3 & (n = 5) \end{cases}$$

We may assume that  $i_X \geq i_Y$ . From (8), we have  $(i_X, i_Y, \mu) = (4, 3, 1)$  provided  $n = 3$ . Hence we see that  $X \cong \mathbb{P}^3$ . Here we take a line on  $X \cong \mathbb{P}^3$  as a base  $\Sigma$  of  $N^2(X)_\mathbb{Q}$ . Then  $d = 1$ . On the other hand, if  $n = 5$ , then we have  $(i_X, i_Y, \mu) = (5, 3, 1)$ . Hence we obtain that  $X \cong Q^5$ . Here  $H^4(X, Q^5) \cong \mathbb{Z}$  and we take its positive generator as a base  $\Sigma$  of  $N^2(X)_\mathbb{Q}$ . Then  $d = 1$ . In both cases, easy calculations imply the following table:

$n$	$i_X$	$d$	$\mu$	$\tau$	$\Delta$	$c_1$	$c_2$
3	4	1	1	2	-4	0	1
5	5	1	1	3	-3	-1	1

Since vector bundles  $\mathcal{N}$  and  $\mathcal{C}$  are determined by their Chern classes among stable bundles (see [13, Lemma 4.3.2] and [14]), hence  $(X, \mathcal{E})$  is isomorphic to  $(\mathbb{P}^3, \mathcal{N})$  or  $(Q^5, \mathcal{C})$ . Then the structure of  $(Y, \mathcal{E}')$  is well-known (for instance, see [15, Proposition 2.6], [11, Example 6.4] and [14, 1.3]). Consequently, Theorem 1.1 holds.  $\square$

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